**NASA Contractor Report** 178200

**ICASE REPORT NO.** 86-72

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(NASA-CR-178200) COMPUTATIONAL METHODS FOR OPTIMAL LINEAR-QUADRATIC COMPENSATORS FOR INFINITE DIMENSIONAL DISCRETE-TIME SYSTEMS Final Report (NASA) 26 p CSCL 72B N87-14056 Unclas G3/64 43929

Contract No. NAS1-18107

October 1986

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA

National Aeronautics and
Space Administration

**Langley Research Center**
Hampton, Virginia 23665

# COMPUTATIONAL METHODS FOR OPTIMAL LINEAR-QUADRATIC COMPENSATORS FOR INFINITE DIMENSIONAL DISCRETE-TIME SYSTEMS+

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**ABSTRACT**

An abstract approximation theory and computational methods are developed for the determination of opitmal linear-quadratic feedback controls, observers and compensators for infinite dimensional discrete-time systems. Particular attention is paid to systems whose open-loop dynamics are described by semigroups of operators on Hilbert spaces. The approach taken is based upon the finite dimensional approximation of the infinite dimensional operator Riccati equations which characterize the optimal feedback control and observer gains. Theoretical convergence results are presented and discussed. Numerical results for an example involving a heat equation with boundary control are presented and used to demonstrate the feasibility of our methods.

* This research was supported in part by the Air Force Office of Scientific Research under Contract No. AFOSR-84-0309.

** This research was supported in part by the Air Force Office of Scientific Research under Contract No. AFOSR-84-0393.

+ Part of this research was carried out while the authors were visiting scientists at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665-5225, which is operated under NASA Contract No. NAS1-18107.

## 1. Introduction

In this paper we develop an approximation theory and computational methods for the determination of the optimal feedback control law for the discrete-time linear-quadratic regulator problem, the optimal state estimator or observer gains and the optimal compensator for infinite dimensional systems. Specifically, we are concerned with systems whose dynamics can be described in terms of linear semigroups of operators on Hilbert spaces. The essential feature of our approach is the finite dimensional approximation of the infinite dimensional operator Riccati equations that characterize the optimal feedback control and observer gains. We develop a general, abstract approximation framework and an associated convergence theory which is applicable to a wide class of problems.

The theory for the discrete-time control problem has been developed previously in [5], while the theory for the observer and compensator in the continuous-time case (with particular emphasis on systems describing the vibration of flexible structures) is treated in [4]. Along with presenting the theory for the discrete-time observer and compensator here for the first time, we briefly review and outline our earlier results for the control problem.

Our treatment below requires that both the discrete-time input and output operators be bounded. As will become evident from the example we present in Section 4 however, an unbounded input operator (i.e. its range is contained in some space larger than the underlying state space) in the continuous-time problem may lead to a bounded input operator when the system is sampled and considered in a discrete-time setting. This notion can, to a certain extent, be generalized to permit the application of the theory we develop here to a wide class of problems which simultaneously involve unbounded input and output operators. This idea will be treated in a forthcoming paper.

Now we provide a brief outline of the remainder of the paper. In Section 2 we present the theory for the optimal infinite dimensional control law, observer and compensator. In Section 3 the approximation theory and convergence results are discussed. An example (including numerical results) involving a heat equation with boundary control is used in Section 4 to demonstrate the feasibility of our methods.

## 2. The Infinite Dimensional Optimal Control Law, State Estimator and Compensator

Let $\{H, \langle\cdot,\cdot\rangle_H\}$ be a Hilbert space and consider the time invariant, discrete-time linear control system

(2.1) $$z_{k+1} = Tz_k + Bu_k, \qquad k = 0,1,2,\ldots.$$
$$z_0 \in H$$

(2.2) $$y_k = Cz_k + Du_k, \qquad k = 0,1,2,\ldots.$$

where $T \in L(H)$, $B \in L(R^m,H)$, $C \in L(H,R^p)$ and $D \in L(R^m,R^p)$. The infinite time horizon discrete-time linear-quadratic regulator (LQR) problem is given by:

Find $u^* = \{u_k^*\}_{k=0}^{\infty} \in \ell_2(0,\infty;R^m)$ which minimizes the quadratic performance index

(2.3) $$J(u) = \sum_{k=0}^{\infty} \langle Qz_k,z_k\rangle_H + u_k^T R u_k$$

where $Q \in L(H)$ is self-adjoint and nonnegative, $R \in L(R^m)$ is a symmetric, positive-definite $m \times m$ matrix, $z_0 \in H$ is given and $z = \{z_k\}_{k=0}^{\infty}$ is determined by the recurrence (2.1).

As in the finite dimensional case the discrete-time system (2.1), (2.2) is frequently the result of sampling a continuous time system of the form

(2.4) $$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \qquad t > 0$$
$$z(0) = z_0$$

(2.5) $$y(t) = Cz(t) + Du(t), \qquad t > 0$$

where $\mathcal{A}$ is the infinitesimal generator of a $C_0$-semigroup of bounded linear operators, $\{\mathcal{T}(t) : t \geq 0\}$, on H and $\mathcal{B} \in \mathcal{L}(R^m, H)$. In this case we have $T = \mathcal{J}(\tau)$ and $B = \int_0^\tau \mathcal{T}(t)\mathcal{B}\, dt$ where $\tau$ is the length of the sampling interval or sampling period.

A control sequence $u \in \ell_2(0,\infty;R^m)$ is said to be admissible for the initial conditions $z_0$ if $J(u) < \infty$. It can be shown (see [5], [10]) that if there is an admissible control for each $z_0 \in H$, then there exists a nonnegative self-adjoint solution $\Pi \in \mathcal{L}(H)$ to the operator Riccati algebraic equation

(2.6) $$\Pi = T^*(\Pi - \Pi B(R + B^*\Pi B)^{-1}B^*\Pi)T + Q.$$

If, in adddition, u admissible for $z_0$ implies $\lim_{k\to\infty} |z_k|_H = 0$ then this solution is unique. Moreover under the two hypotheses given above, the LQR problem admits a unique solution $u^*$ for each $z_0 \in H$ with $J(u^*) = \langle \Pi z_0, z_0 \rangle_H$. The optimal control is given in feedback form by

$$u_k^* = -Fz_k^*, \qquad k = 0,1,2,\ldots$$

where the optimal feedback control gains F are given by

$$(2.7)\qquad F = (R + B^*\Pi B)^{-1}B^*\Pi T$$

and $z^* = \{z_k^*\}_{k=0}^{\infty}$ is the resulting optimal state trajectory. We have

$$z_{k+1}^* = Sz_k^*, \qquad k = 0,1,2,\ldots$$
$$z_0^* = z_0$$

with the optimal closed-loop state transition operator S given by

$$(2.8)\qquad S = T - BF.$$

If Q is also coercive (i.e., $Q \geq \alpha$ for some $\alpha > 0$) then S has spectral radius less than one and S is uniformly exponentially stable with

$$|S^k| \leq (|\Pi|/\alpha)(1 - \alpha/|\Pi|)^k, \qquad k = 0,1,2,\ldots .$$

For each $j = 1,2,\ldots,m$ an application of the Riesz Representation Theorem yields that the $j^{th}$ component of the optimal $k^{th}$ control input is given by

$$[u_k^*]_j = - \langle f_j, z_k^* \rangle_H$$

for some $f_j \in H$. The vector $f = (f_1,f_2,\ldots,f_m)^T \in \times_{j=1}^{m} H$ is referred to as the optimal functional feedback control gains.

In order to implement a feedback control law of the form $u_k = -Gz_k$, $k = 0,1,2,\ldots$ where $G \in L(H,R^m)$ it is necessary that the full infinite dimensional state $z_k$ be available for each k. In practice, however, only a finite dimensional observation $y_k \varepsilon R^p$ of the state, as given in (2.2), is provided. Consequently, a state estimator or observer is required.

For any operator $\hat{G} \varepsilon L(R^p,H)$ the discrete-time linear system

$$(2.9) \qquad \hat{z}_{k+1} = T\hat{z}_k + Bu_k + \hat{G}\{y_k - C\hat{z}_k - Du_k\}, \qquad k = 0,1,2,\ldots$$
$$\hat{z}_0 \varepsilon H$$

is called an observer or estimator for the system (2.1), (2.2). The feedback control law

$$(2.10) \qquad u_k = -G\hat{z}_k, \qquad k = 0,1,2,..$$

along with the observer (2.9) is referred to as a compensator for the system (2.1), (2.2).

If we define $e_k = z_k - \hat{z}_k$, $k = 0,1,2,\ldots$ then direct calculation yields

$$e_{k+1} = (T - \hat{G}C)e_k, \qquad k = 0,1,2,\ldots$$

or $e_k = \hat{S}(\hat{G})^k e_0$, $k = 0,1,2,\ldots$ where $\hat{S}(\hat{G}) \varepsilon L(H)$ is given by

$$(2.11) \qquad \hat{S}(\hat{G}) = T - \hat{G}C.$$

If the control law or compensator (2.10) is to be used, then it is desirable to have $e_k \to 0$ as $k \to \infty$. The observer corresponding to $\hat{G}$ is said to be strongly stable if $|\hat{S}(\hat{G})^k \hat{z}_0|_H \to 0$ as $k \to \infty$ for each $\hat{z}_0 \varepsilon H$. It is said to be uniformly exponentially stable if there exist positive constants $\hat{M}$ and $\hat{r}$ with $\hat{r} < 1$ such that

$$|\hat{S}(\hat{G})^k| \leq \hat{M}\hat{r}^k, \qquad k = 0,1,2,\ldots .$$

If $z = \{z_k\}_{k=0}^{\infty}$ and $\hat{z} = \{\hat{z}_k\}_{k=0}^{\infty}$ are generated by (2.1) and (2.9) respectively with $u = \{u_k\}_{k=0}^{\infty}$ given by (2.10) then $z = \{z_k\}_{k=0}^{\infty} = \{(z_k,\hat{z}_k)^T\}_{k=0}^{\infty}$ satisfies the recurrence

$$(2.12) \qquad z_{k+1} = S(G,\hat{G})\, z_k, \qquad k = 0,1,2,\ldots$$

where $S(G,\hat{G}) \in L(H \times H)$ is given by

$$S(G,\hat{G}) = \begin{bmatrix} T & -BG \\ \hat{G}C & T - BG - \hat{G}C \end{bmatrix}.$$

The system (2.12) or equivalently

$$z_k = S(G,\hat{G})^k z_0, \qquad k = 0,1,2,\ldots$$

is the closed-loop system corresponding to the control system (2.1), (2.2), the observer (2.9) and the compensator (2.10).

We recall (2.8) and by analogy to (2.11), for $G \in L(H,R^m)$ we adopt the notation

$$(2.13) \qquad S(G) = T - BG.$$

Using the facts that

$$z_{k+1} = Tz_k - BG\hat{z}_k$$
$$= (T - BG)z_k + BGe_k$$
$$= S(G)z_k + BGe_k$$

and

$$e_{k+1} = [I, -I]\ z_{k+1} = [I, -I]\ \mathcal{S}(G,\hat{G})\ z_k$$
$$= \hat{S}(\hat{G})[I, -I]\ z_k$$
$$= \hat{S}(\hat{G})e_k$$

and consequently that

$$\mathcal{S}(G,\hat{G}) = U \begin{bmatrix} S(G) & BG \\ 0 & \hat{S}(G) \end{bmatrix} U^{-1}$$

where $U = U^{-1} \in \mathcal{L}(H \times H)$ is given by

$$U = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix},$$

standard arguments can be used to establish the following result.

<u>Theorem</u> Suppose that there exist positive constants $M$, $\hat{M}$, $r$ and $\hat{r}$ for which

$$|S(G)^k| < Mr^k \text{ and } |\hat{S}(\hat{G})^k| < \hat{M}\hat{r}^k, \qquad k = 0,1,2,\ldots .$$

Then for each $\mathcal{r}$ with $\mathcal{r} > \max(r,\hat{r})$ there exists a positive constant $\mathcal{M}$ such that

$$|\mathcal{S}(G,\hat{G})^k| < \mathcal{M}\mathcal{r}^k, \qquad k = 0,1,2,\ldots .$$

In particular if $S(G)$ and $\hat{S}(\hat{G})$ are uniformly exponentially stable (i.e. $r$, $\hat{r} < 1$) then so too is $\mathcal{S}(G,\hat{G})$. Also, the spectrum of $\mathcal{S}(G,\hat{G})$, $\sigma(\mathcal{S}(G,\hat{G}))$ is given by

$$\sigma(\mathcal{S}(G,\hat{G})) = \sigma(S(G)) \cup \sigma(\hat{S}(\hat{G})).$$

By analogy to the finite dimensional case (see [7]) we define the optimal discrete-time observer for the system (2.1), (2.2) to be the system (2.9) with the observer gains $\hat{G}$ replaced by $\hat{F}$ given by

$$\hat{F} = T\hat{\Pi}C^*(\hat{R} + C\hat{\Pi}C^*)^{-1}$$

where $\hat{\Pi} \in \mathcal{L}(H)$ is the minimal nonnegative self-adjoint solution (if one exists) to the Riccati algebraic equation

$$\hat{\Pi} = T(\hat{\Pi} - \hat{\Pi}C^*(\hat{R} + C\hat{\Pi}C^*)^{-1}C\hat{\Pi})T^* + \hat{Q} , \tag{2.14}$$

$\hat{Q} \in \mathcal{L}(H)$ is nonnegative self-adjoint and $\hat{R} \in \mathcal{L}(R^p)$ is a symmetric positive definite p×p matrix. When G in (2.10) is taken to be the optimal feedback control gains F given in (2.7) and $\hat{z} = \{\hat{z}_k\}_{k=0}^{\infty}$ is taken to be $\hat{z}^* = \{\hat{z}_k^*\}_{k=0}^{\infty}$, the trajectory determined by the optimal observer, the resulting feedback control law

$$\hat{u}_k^* = -F\hat{z}_k^* , \qquad k = 0,1,2,.. \tag{2.15}$$

is known as the optimal infinite dimensional compensator. The optimal closed-loop system is given by

$$z^*_{k+1} = \mathcal{S}z^*_k = \mathcal{S}(F,\hat{F})\, z^*_k\,, \qquad k = 0,1,2,\ldots\,.$$

We note that the adjoint of the optimal observer gains $\hat{F}$ are the optimal control gains for the linear regulator problem obtained by replacing the operators T and B in (2.1) with the operators $T^*$ and $C^*$ and the operators Q and R in (2.3) with the operators $\hat{Q}$ and $\hat{R}$. Consequently the necessary and sufficient conditions for the existence of a nonnegative self-adjoint solution (and therefore a minimal nonnegative self-adjoint solution) $\hat{\Pi}$ to the Riccati algebraic equation (2.14) become clear and can be found in [5]. In addition, it is now also easy to specify the conditions under which 1) (2.14) has a unique nonnegative self-adjoint solution and 2) the operator

$$\hat{S} = \hat{S}(\hat{F}) = T - \hat{F}C$$

will be uniformly exponentially stable.

The optimal observer gains $\hat{F}$ is an element in $\mathcal{L}(R^p,H)$. They therefore have a representation of the form

$$\hat{F}y = \sum_{i=1}^{p} \hat{f}_i y_i\,, \quad y = (y_1,\, y_2,\ldots,y_p)^T \in R^p$$

where $\hat{f}_i \in H$, $i = 1,2,\ldots,p$. The vector $\hat{f} = (\hat{f}_1,\hat{f}_2,\ldots,\hat{f}_p)^T \in \overset{p}{\underset{1}{\times}} H$ is referred to as the optimal functional observer gains.

Ordinarily of course, the otpimal observer has a stochastic interpretation. The optimal observer as we have defined it above is the natural extension to infinite dimensions of the well known finite dimensional discrete-time Kalman-Bucy filter for the case in which the state and output equations are corrupted by uncorrelated, stationary Gaussian white noise processes with zero mean and respective covariance operators (matrices) $\hat{Q}$ and $\hat{R}$ (see [7]). Under appropriate additional hypotheses (i.e. that the state weighting operator Q in (2.3) is trace class, see [1]) it is also possible to provide the optimal infinite dimensional compensator given by (2.15) with the standard finite dimensional stochastic interpretation (i.e. as the usual optimal LQG compensator, see [3], [4]). The ideas which have been presented above and the approximation theory which will be described in the next section require only that the conditions which have been set forth thus far hold. We shall therefore continue to take a strictly deterministic approach and assume that the operators $\hat{Q}$ and $\hat{R}$ (as well as Q and R) are determined by engineering criteria (for example, stability margins, robustness of the closed-loop systems, etc.) rather than via an assumed noise model incorporated into the underlying dynamics.

## 3. Approximation and Convergence

In this section we develop an approximation framework which yields finite dimensional approximations to the optimal infinite dimensional control and observer gains and the optimal infinite dimensional compensator. Central to our approach are finite dimensional approximations to the infinite dimensional operator Riccati equations (2.6) and (2.14). The approximating equations can be solved using conventional techniques (for example, eigenvalue-eigenvector or Schur decomposition of the associated Hamiltonian matrices). Under mild and rather general assumptions, the convergence of the approximation can be argued.

For each N = 1,2,... let $H_N$ be a finite dimensional subspace of H. Let $P_N : H \to H_N$ denote the orthogonal projection of H onto $H_N$ with respect to the H-inner product, $\langle\cdot,\cdot\rangle_H$. For $T_N \in \mathcal{L}(H_N)$, $B_N \in \mathcal{L}(R^m,H_N)$ and $Q_N \in \mathcal{L}(H_N)$ with $Q_N$ nonnegative self-adjoint, we consider the Riccati algebraic equation

$$(3.1)\qquad \Pi_N = T_N^*(\Pi_N - \Pi_N B_N(R + B_N^*\Pi_N B_N)^{-1}B_N^*\Pi_N)T_N + Q_N .$$

We assume that for each N, the equation (3.1) has a unique nonnegative self-adjoint solution $\Pi_N \in L(H_N)$ and define the $N^{th}$ approximating optimal control gains by

$$F_N = (R + B_N^*\Pi_N B_N)^{-1}B_N^*\Pi_N T_N .$$

For the estimator, we take $\hat{Q}_N \in L(H_N)$ nonnegative self-adjoint and $C_N \in L(H_N, R^p)$ and consider the equation

$$(3.2)\qquad \hat{\Pi}_N = T_N(\hat{\Pi}_N - \hat{\Pi}_N C_N^*(\hat{R} + C_N\hat{\Pi}_N C_N^*)^{-1}C_N\hat{\Pi}_N)T_N^* + \hat{Q}_N .$$

Assuming a unique nonnegative self-adjoint solution $\hat{\Pi}_N \in L(H_N)$, we define the $N^{th}$ approximating optimal observer gains by

$$\hat{F}_N = T_N\hat{\Pi}_N C_N^*(\hat{R} + C_N\hat{\Pi}_N C_N^*)^{-1} .$$

The $N^{th}$ approximating optimal compensator is given by

$$(3.3)\qquad \hat{u}^*_{N,k} = -F_N\hat{z}^*_{N,k}, \qquad k = 0,1,2,\ldots$$

where $\hat{z}^*_N = \{\hat{z}^*_{N,k}\}_{k=0}^{\infty}$ is determined from the $N^{th}$ approximating optimal observer

$$(3.4)\qquad \hat{z}^*_{N,k+1} = T_N\hat{z}^*_{N,k} + B_N\hat{u}^*_{N,k} + \hat{F}_N\{y^*_{N,k} - C_N\hat{z}^*_{N,k} - D\hat{u}^*_{N,k}\}, \qquad k = 0,1,2,\ldots$$

$$\hat{z}^*_{N,0} = P_N\hat{z}_0 \in H_N .$$

The output sequence $y_N^* = \{y_{N,k}^*\}_{k=0}^{\infty}$ is given by

$$y_{N,k}^* = Cz_{N,k}^* + D\hat{u}_{N,k}^* \qquad k = 0,1,2,\ldots$$

where

$$z_{N,k+1}^* = Tz_{N,k}^* - BF_N\hat{z}_{N,k}^* , \qquad k = 0,1,2,\ldots$$
$$z_{N,0}^* = z_0 .$$

The $N^{th}$ approximating optimal closed-loop sytem evolves according to the recurrence

$$\mathcal{Z}_{N,k+1}^* = \mathcal{S}_N \mathcal{Z}_{N,k}^*, \qquad k = 0,1,2,\ldots$$

where $\mathcal{Z}_{N,k}^* = (z_{N,k}^*, \hat{z}_{N,k}^*)^T$ and $\mathcal{S}_N \varepsilon \mathcal{L}(H \times H_N)$ is given by

$$\mathcal{S}_N = \begin{bmatrix} T & -BF_N \\ \hat{F}_N C & T_N - B_N F_N - \hat{F}_N C_N \end{bmatrix} .$$

The equations and formulas given above are operator equations and as such can not be used in computations directly. It is their matrix equivalents with respect to a given basis for $H_N$ that are required.

We assume that the collection $\{\phi_i^N\}_{i=1}^{K_N}$ is a (not necessarily orthogonal) basis for $H_N$ and define $\Phi^N \varepsilon \times_1^{K_N} H_N$ by $\Phi^N = (\phi_1^N, \phi_2^N, \ldots, \phi_{K_N}^N)^T$. We adopt the notational convention that for a linear operator L with domain and range in $H_N$, $R^m$

or $R^p$, its matrix representation with respect to the basis $\{\phi_i^N\}_{i=1}^{K_N}$ for $H_N$ and the standard bases for $R^m$ and $R^p$ will be denoted by [L].

If we define the Gram matrix $M^N = \langle\Phi^N,(\Phi^N)^T\rangle_H$ then $[T_N^*] = (M^N)^{-1}[T_N]^T M^N$, $[B_N^*] = [B_N]^T M^N$ and $[C_N^*] = (M^N)^{-1}[C_N]^T$. Also $\Gamma^N = M^N[\Pi_N]$ and $\hat{\Gamma}^N = [\hat{\Pi}_N](M^N)^{-1}$ are respectively, the unique nonnegative symmetric solutions to the matrix Riccati algebraic equations

$$(3.5)\qquad \Gamma^N = [T_N]^T(\Gamma^N - \Gamma^N[B_N](R + [B_N]^T\Gamma^N[B_N])^{-1}[B_N]^T\,\Gamma^N)[T_N] + Q^N\ ,$$

and

$$(3.6)\qquad \hat{\Gamma}^N = [T_N](\hat{\Gamma}^N - \hat{\Gamma}^N[C_N]^T(\hat{R} + [C_N]\hat{\Gamma}^N[C_N]^T)^{-1}[C_N]\hat{\Gamma}^N)[T]^T + \hat{Q}^N$$

where $Q^N = M^N[Q_N]$ and $\hat{Q}^N = [\hat{Q}_N](M^N)^{-1}$ . The matrix representation for the approximating optimal control gains is given by

$$[F_N] = (R + [B_N]^T\Gamma^N[B_N])^{-1}[B_N]^T\Gamma^N[T_N]$$

and for the approximating optimal observer gains by

$$[\hat{F}_N] = [T_N]\hat{\Gamma}^N[C_N]^T(\hat{R} + [C_N]\hat{\Gamma}^N[C_N]^T)^{-1}\ .$$

If we write $\hat{z}^*_{N,k} = (\Phi^N)^T\hat{\zeta}^*_{N,k}$ with $\hat{\zeta}^*_{N,k} \in R^{K_N}$ , $k = 0,1,2,\ldots$ then from (3.3) we obtain

$$\hat{u}^*_{N,k} = -[F_N]\,\hat{\zeta}^*_{N,k},\qquad k = 0,1,2,\ldots$$

and from (3.4)

$$\hat{\zeta}^*_{N,k+1} = [T_N]\hat{\zeta}^*_{N,k} + [B_N]\hat{u}^*_{N,k} + [\hat{F}_N]\{y^*_{N,k} - [C_N]\,\hat{\zeta}^*_{N,k} - D\hat{u}^*_{N,K}\}, \quad k = 0,1,2,\ldots$$

$$\hat{\zeta}^*_{N,0} = (M^N)^{-1}\ \langle\Phi^N,\hat{z}_0\rangle_H \ .$$

If we let $f^N = (f^N_1,\ f^N_2,\ldots,f^N_m)^T \ \varepsilon \ \overset{m}{\underset{1}{\times}} H_N$ denote the $N^{th}$ approximating optimal functional feedback control gains, then from

$$F_N z_N = \langle f^N, z_N\rangle_H = [F_N]\zeta_N$$

where $z_N = (\Phi^N)^T\zeta_N \ \varepsilon \ H_N$, we find

$$f^N = [F_N](M^N)^{-1}\Phi^N \ .$$

Similarly, the $N^{th}$ approximating optimal functional observer gains $\hat{f}^N = (\hat{f}^N_1,\hat{f}^N_2,\ldots,\hat{f}^N_p)^T \ \varepsilon \ \overset{p}{\underset{1}{\times}} H_N$ are given by

$$\hat{f}^N = [\hat{F}_N]^T\Phi^N \ .$$

It is immediately clear that the limiting behavior of the approximation (i.e. as $N \to \infty$) is determined by the limiting behavior of the solutions to the finite dimensional Riccati equations (3.1) and (3.2). A convergence theory for approximations to discrete-time Riccati equations was developed in detail in [5]. We briefly summarize those results here.

We assume that the spaces $H_N$ are H-approximating in the sense that the projections $P_N$ converge strongly to the identity on H as $N \to \infty$. Also, we require that $T_N P_N z \to Tz$, $T^*_N P_N z \to T^* z$, $Q_N P_N z \to Qz$ and $\hat{Q}_N P_N \to \hat{Q}z$ for each $z \ \varepsilon \ H$

and that $B_N \to B$ and $C_N P_N \to C$ in norm as $N \to \infty$. Define $S_N$, $\hat{S}_N \in L(H_N)$ by $S_N = T_N - B_N F_N$ and $\hat{S}_N = T_N - \hat{F}_N C_N$.

If there exists a positive constant $M$ $(\hat{M})$ independent of $N$ for which $\Pi_N < M$ $(\hat{\Pi}_N < \hat{M})$ then there exists a nonnegative self-adjoint solution $\Pi$ $(\hat{\Pi})$ of (2.6) ((2.4)) and $\Pi_N P_N$ $(\hat{\Pi}_N P_N)$ converges weakly to $\Pi$ $(\hat{\Pi})$ as $N \to \infty$. If, in addition, there exists a positive constant $r$ $(\hat{r})$ less than one and independent of $N$ for which

$$(3.7) \qquad |S_N^k| < M\, r^k \qquad (|\hat{S}_N^k| < \hat{M}\, \hat{r}^k), \qquad k = 1,2,\ldots$$

then $\Pi_N P_N$ $(\hat{\Pi}_N P_N)$ will converge strongly to $\Pi$ $(\hat{\Pi})$.

If the operators $Q_N$ $(\hat{Q}_N)$ are uniformly (with repsect to N) coercive and the $\Pi_N$ $(\hat{\Pi}_N)$ are uniformly bounded then there exists a positive $r$ $(\hat{r})$ less than one for which (3.7) holds. If it is also true that $Q$ $(\hat{Q})$ is trace class and $Q_N P_N$ $(\hat{Q}_N P_N)$ converges in trace norm to $Q$ $(\hat{Q})$, then $\Pi$ $(\hat{\Pi})$ is also trace class and $\Pi_N P_N$ $(\hat{\Pi}_N P_N)$ converges in trace norm to $\Pi$ $(\hat{\Pi})$.

The consequences of these results in the context of the control and observer problems are at once clear. If $\Pi_N P_N \to \Pi$ weakly as $N \to \infty$, then $F_N P_N \to F$ and $S_N P_N \to S$ strongly and $f_i^N \to f_i$, $i = 1,2,\ldots,m$ weakly in $H$ as $N \to \infty$. If $\Pi_N P_N \to \Pi$ strongly, then $F_N \to F$ in norm, $S_N P_N \to S$ strongly and $f_i^N \to f_i$, $i = 1,2,\ldots,m$ strongly in $H$ as $N \to \infty$. For the observer problem, if $\hat{\Pi}_N P_N \to \hat{\Pi}$ weakly, then $\hat{F}_N \to \hat{F}$ and $\hat{S}_N P_N \to \hat{S}$ weakly and $\hat{f}_i^N \to \hat{f}_i$, $i = 1,2,\ldots,p$ weakly in $H$ as $N \to \infty$. If $\hat{\Pi}_N P_N \to \hat{\Pi}$ strongly, then $\hat{F}_N \to \hat{F}$ in norm, $\hat{S}_N P_N \to \hat{S}$ strongly and $\hat{f}_i^N \to \hat{f}_i$, $i = 1,2,\ldots,p$ strongly in $H$ as $N \to \infty$.

Let $\mathcal{P}_N$ denote the projection of $H \times H$ onto $H \times H_N$ defined by $\mathcal{P}_N(z_1,z_2) = (z_1,P_N z_2)$. If $\Pi_N P_N \to \Pi$ weakly or strongly then $\mathcal{S}_N \mathcal{P}_N \to \mathcal{S}$ weakly or strongly depending only upon whether $\hat{\Pi}_N P_N \to \hat{\Pi}$ weakly or strongly as $N \to \infty$. Under appropriate additional hypotheses on the spectral properties of the open loop system and the nature of the approximation spaces $H_N$ and the projections $P_N$, it is possible to obtain a result regarding the norm convergence of $\mathcal{S}_N \, P_N$ to $\mathcal{S}$ (see [4]). An important consequence of this norm convergence of the closed-loop systems is that the uniform exponential stability of $\mathcal{S}$ would imply the uniform exponential stability of $\mathcal{S}_N$ for all N sufficiently large.

**Remark** It is sometimes the case that while $T_N P_N \to T$ strongly, $T_N^* P_N \to T^*$ only weakly (see, for example [2]). In this instance it remains possible to demonstrate the weak convergence of $\Pi_N P_N$ to $\Pi$ (see [5]). However, we do not see how to retain the weak convergence of $\hat{\Pi}_N P_N$ to $\hat{\Pi}$.

**Remark** If the discrete-time system (2.1), (2.2) was obtained via the sampling of a continuous time system of the form (2.4), (2.5) then the approximation to T, $T_N$, is frequently obtained by approximating the operator $A$ by an operator $A_N$ and then setting $T_N = \exp(A_N \tau)$. The convergence of $T_N P_N$ to T and $T_N^* P_N$ to $T^*$ can then be argued using the well known Trotter-Kato semigroup approximation result (see [6]).

## 4. An Example: A Heat Equation with Boundary Input

We consider the parabolic system with boundary control given by

$$\frac{\partial w}{\partial t}(t,x) = \frac{\partial}{\partial x} a(x) \frac{\partial w}{\partial x}(t,x), \qquad t > 0, \; x \in (0,1)$$

$$w(t,0) = 0 \qquad w(t,1) = v(t), \qquad t > 0$$

$$w(0,x) = \phi(x), \qquad x \in [0,1]$$

where $a \in H^1(0,1)$, $a(x) > 0$, $x \in [0,1]$, $\phi \in L_2(0,1)$ and $v \in L_2(0,\infty)$. We take the average temperature over an interval of small, but positive, length,

$[\varepsilon_1, \varepsilon_2]$. That is

$$y(t) = \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} w(t,x)dx, \qquad 0 < \varepsilon_1 < \varepsilon_2 < 1.$$

We choose the state space H to be $L_2(0,1)$ endowed with the usual inner product and denote the length of the sampling interval by $\tau$. We consider piecewise constant controls of the form

$$v(t) = u_k, \quad t \in [k\tau,(k+1)\tau), \quad k = 0,1,2,\ldots$$

and take the discrete-time state $z_k \in L_2(0,1)$ to be

$$z_k = \lim_{t \to k\tau^-} w(t,\cdot), \qquad k = 1,2,\ldots$$
$$z_0 = \phi \ .$$

The resulting discrete-time control system is given by (see [5])

$$z_{k+1} = Tz_k + Bu_k, \qquad k = 0,1,2,\ldots$$
$$z_0 = \phi$$

$$y_k = Cz_k, \qquad k = 0,1,2,\ldots \ .$$

The open-loop state transition operator T is given by $T = \mathcal{T}(\tau)$ where $\{\mathcal{T}(t) : t \geq 0\}$ is the analytic semigroup of bounded linear operators on H with infinitesimal generator $A$ defined by $A\psi = (a\psi')'$ for $\psi \in H^2(0,1) \cap H_0^1(0,1)$. The input operator $B \in \mathcal{L}(R,H) = H$ is given by

$$B = ((I - T(\tau))\psi_0 + \int_0^\tau T(\sigma)a'd\sigma$$

where $\psi_0 \in H$ is given by $\psi_0(x) = x$, $x \in [0,1]$. The output operator $C \in L(H,R) = H'$ takes the form

$$C\psi = \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \psi(x)dx, \qquad \psi \in L_2(0,1).$$

The performance index for the control problem is assumed to be of the form

$$J(u) = \sum_{k=0}^{\infty} q|z_k|_H^2 + ru_k^2$$

with $q \geq 0$ and $r > 0$. For the optimal observer problem we assume that $\hat{q} \geq 0$ and $\hat{r} > 0$ are given.

The operator $A$ is densely defined, self-adjoint and has compact resolvent. It satisfies the dissipative inequality

$$(4.1) \qquad \langle A\psi,\psi\rangle_H \leq -\omega|\psi|_H^2, \quad \psi \in H^2(0,1) \cap H_0^1(0,1)$$

for some $\omega > 0$ and consequently the semigroup $\{T(t) : t \geq 0\}$ is uniformly exponentially stable with $|T(t)| \leq e^{-\omega t}$, $t \geq 0$.

It follows therefore, that both the optimal control and observer problems (along with the associated operator Riccati equations) have unique solutions. The optimal control law is given by

$$u_k^* = -Fz_k^* = -\langle f,z_k^*\rangle_H = -\int_0^1 f(x)z_k^*(x)dx, \qquad k = 0,1,2,\ldots$$

where $f \in L_2(0,1)$ is the optimal functional feedback control gain. The optimal observer gains have the form

$$\hat{F}y = \hat{f}y, \qquad y \in R$$

where $\hat{f} \in L_2(0,1)$ is the optimal functional observer gain.

We note that if $a(x) = a$, a constant, and $\varepsilon_1$ and $\varepsilon_2$ are chosen appropriately, then all of the open-loop modes will be controllable and observable (i.e. $B$ and $C^*$ are not orthogonal to any of the eigenfunctions of $T = T^*$).

We use a standard Ritz-Galerkin approach to define a linear spline based approximation scheme. For each $N = 2,3,\ldots$ let $\{\phi_j^N\}_{j=1}^{N-1}$ denote the usual "hat" functions on $[0,1]$ which vanish on the boundary. They are given by

$$\phi_j^N(x) = \begin{cases} Nx - j + 1 & x \in [\frac{j-1}{N}, \frac{j}{N}] \\ j + 1 - Nx & x \in [\frac{j}{N}, \frac{j+1}{N}] \\ 0 & \text{elsewhere} \end{cases}$$

for $j = 1,2,\ldots,N-1$.

Letting $V$ be the space $H_0^1(0,1)$ endowed with the inner product $\langle\psi_1,\psi_2\rangle_V = \langle a\psi_1', \psi_2'\rangle_H$, the usual compact embeddings $V \subset H \subset V'$ hold. Set $H_N = \text{span}\,\{\phi_j^N\}_{j=1}^{N-1}$ and denote by $P_N$ the orthogonal projection of $H$ onto $H_N$ with respect to the $H$-inner product. Note that $H_N \subset V$ and denote by $P_N^V$ the orthogonal projection of $V$ onto $H_N$ with respect to the $V$ inner product.

From (4.1) we find $0 \in \rho(A)$. Consequently $A^{-1}$ exists and is compact. Define $A_N \in L(H_N)$ to be the inverse of the operator $A_N^{-1} = P_N^V A^{-1}|_{H_N}$. It is not difficult to show (see [5]) that the operator $A_N$ is well defined, self-adjoint and satisfies

$$\langle A_N\psi_N,\psi_N\rangle_H \le -\omega|\psi_N|_H^2, \qquad \psi_N \in H_N .$$

Setting $T_N(t) = \exp(A_N t)$, the semigroups of bounded linear operators on $H_N$, $\{T_N(t) : t \ge 0\}$ are uniformly exponentially stable with $|T_N(t)| \le e^{-\omega t}$, $t \ge 0$.

Elementary properties of linear spline functions (see [9]) imply that $P_N \to I$ strongly on H and $P_N^V \to I$ strongly on V as $N \to \infty$. Since $A^{-1}$ is compact and

$$|P_N^V A^{-1}\psi - A^{-1}\psi|_H \le |P_N^V A^{-1}\psi - A^{-1}\psi|_V = |(P_N^V - I)A^{-1}\psi|_V, \qquad \psi \in H$$

we conclude that $P_N^V A^{-1} \to A^{-1}$ in norm as $N \to \infty$. It follows that $A_N^{-1}P_N \to A^{-1}$ strongly as $N \to \infty$ and therefore (using the Trotter-Kato semigroup approximation theorem, see [6]) that $T_N(t)P_N\psi \to T(t)\psi$ and $T_N^*(t)P_N\psi \to T^*(t)\psi$ as $N \to \infty$ for each $\psi \in H$ uniformly on bounded t-intervals.

Setting $T_N = T_N(\tau)$, $Q_N = qP_N$, $\hat{Q}_N = \hat{q}P_N$, $C_N = CP_N$ and

$$B_N = (I - T_N(\tau))P_N\psi_0 + \int_0^\tau T_N(\sigma)P_N a'd\sigma,$$

the uniform exponential stability of $T_N$ implies that the $N^{th}$ approximating Riccati equations (3.1) and (3.2) have unique, nonnegative, self-adjoint solutions $\Pi_N$ and $\hat{\Pi}_N$ for each N and that $\Pi_N P_N \to \Pi$ and $\hat{\Pi}_N P_N \to \hat{\Pi}$ strongly as $N \to \infty$. For $\psi \in H$ and $y \in R$ we have that the $N^{th}$ approximating optimal feedback control and observer gains $F_N$ and $\hat{F}_N$ satisfy

$$F_N P_N \psi = \langle f^N, P_N\psi\rangle_H = \langle f^N,\psi\rangle_H = \int_0^1 f^N(x)\psi(x)dx$$

and

$$\hat{F}_N y = \hat{f}^N y$$

for some $f^N$, $\hat{f}^N \varepsilon H_N$ with $f^N \to f$ and $\hat{f}^N \to \hat{f}$ strongly in $L_2(0,1)$ as $N \to \infty$.

Recalling the definition of $\Phi^N$ and $M^N$, the $(N-1) \times (N-1)$ matrix representation for the operator $A_N$ is given by $[A_N] = (M^N)^{-1}(L^N)$ where $L^N = -\langle\Phi^N,(\Phi^N)^T\rangle_V$ . Then $[T_N] = \exp([A_N]\tau)$ and defining $\psi_0^N = \langle\Phi^N,\psi_0\rangle_H$ , $a_N' = \langle\Phi^N,a'\rangle_H$ and $I^N$ to be the $(N-1) \times (N-1)$ identity matrix we have

$$[B_N] = (I^N - [T_N])(M^N)^{-1}\psi_0^N + \int_0^\tau \exp([A_N]\sigma)(M^N)^{-1}a_N' d\sigma$$

$$= (I^N - [T_N])(M^N)^{-1}\psi_0^N + [A_N]^{-1}([T_N] - I^N)(M^N)^{-1}a_N' ,$$

$[Q_N] = qI^N$, $[\hat{Q}_N] = \hat{q}I^N$ and $[C_N] = C(\Phi^N)^T$ with $f^N = [F_N](M^N)^{-1}\Phi^N$ and $\hat{f}^N = [F_N]^T\Phi^N$ .

Setting $a(x) = 1$, $x \varepsilon [0,1]$, $q = 1$, $\hat{q} = 1$, $r = 1$, $\hat{r} = 1$, $\tau = .01$, $\varepsilon_1 = \frac{1}{2} - .04\sqrt{2}$ and $\varepsilon_2 = \frac{1}{2} + .03\sqrt{2}$ we used our scheme to obtain the approximating functional gains $f^N$ and $\hat{f}^N$ for various values of N plotted in Figures 4.1 and 4.2 respectively below. The matrix Riccati equations (3.5) and (3.6) were solved using a generalized eigenvector approach (see [8]). All computations were carried out on an IBM PC personal computer.

**Acknowledgement** The authors would like to gratefully acknowledge the assistance provided by Mr. Milton Lie in carrying out the computations reported on in Section 4.

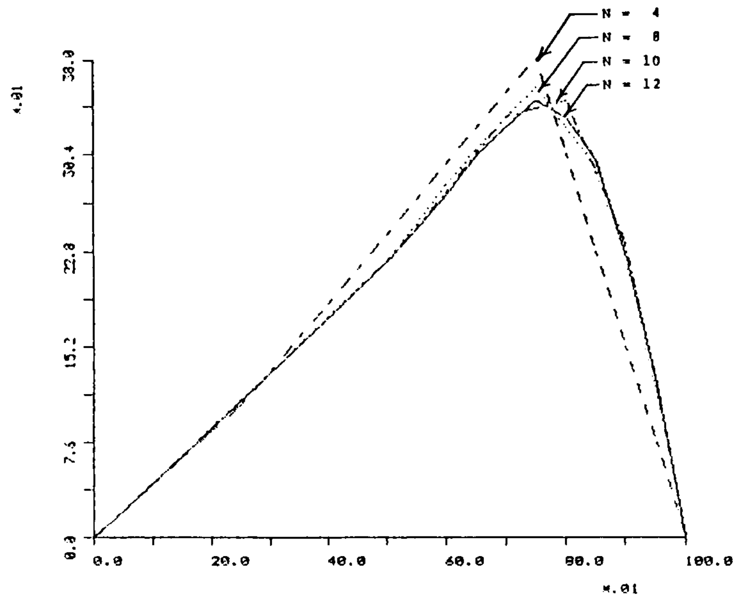


Figure 4.1: Approximating optimal functional feedback control gains, $f^N$.

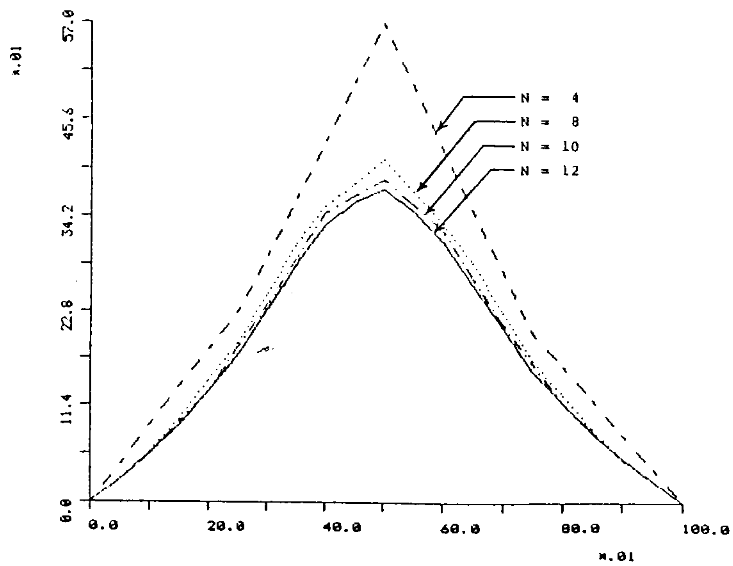


Figure 4.2: Approximating optimal functional observer gains, $\hat{f}^N$.

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Standard Bibliographic Page

| 1. Report No. NASA CR-178200 ICASE Report No. 86-72 | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| 4. Title and Subtitle COMPUTATIONAL METHODS FOR OPTIMAL LINEAR-QUADRATIC COMPENSATORS FOR INFINITE DIMENSIONAL DISCRETE-TIME SYSTEMS | | 5. Report Date October 1986 |
| | | 6. Performing Organization Code |
| 7. Author(s) J. S. Gibson, I. G. Rosen | | 8. Performing Organization Report No. 86-72 |
| 9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225 | | 10. Work Unit No. |
| | | 11. Contract or Grant No. NAS1-18107 |
| 12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546 | | 13. Type of Report and Period Covered Contractor Report |
| | | 14. Sponsoring Agency Code 505-90-21-01 |

15. Supplementary Notes

Langley Technical Monitor:
J. C. South

Final Report

Submitted to Proc. of the Conf. on the Control and Identification of Distributed Paramter Systems, Vorau Austria

16. Abstract

An abstract approximation theory and computational methods are developed for the determination of opitmal linear-quadratic feedback controls, observers and compensators for infinite dimensional discrete-time systems. Particular attention is paid to systems whose open-loop dynamics are described by semigroups of operators on Hilbert spaces. The approach taken is based upon the finite dimensional approximation of the infinite dimensional operator Riccati equations which characterize the optimal feedback control and observer gains. Theoretical convergence results are presented and discussed. Numerical results for an example involving a heat equation with boundary control are presented and used to demonstrate the feasibility of our methods.

| 17. Key Words (Suggested by Authors(s)) | 18. Distribution Statement |
|---|---|
| infinite dimensional systems, feedback control, state estimation, compensator, approximation | 64 - Numerical Analysis<br>66 - Systems Analysis<br><br>Unclassified - unlimited |

| 19. Security Classif.(of this report) | 20. Security Classif.(of this page) | 21. No. of Pages | 22. Price |
|---|---|---|---|
| Unclassified | Unclassified | 25 | A02 |

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1986